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# SYLOW-METACYCLIC GROUPS AND Q-ADMISSIBILITY

BY

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ABSTRACT

A finite group $G$ is **Q**-*admissible* if there exists a division algebra finite dimensional and central over **Q** which is a crossed product for $G$. A **Q**-admissible group is necessarily Sylow-metacyclic (all its Sylow subgroups are metacyclic). By means of an investigation into the structure of Sylow-metacyclic groups, the inverse problem (is every Sylow-metacyclic group **Q**-admissible?) is essentially reduced to groups of order $2^a 3^b$ and to a list of known "almost simple" groups.

A finite group $G$ is called **Q**-*admissible* [13] if there exists a division algebra $D$ finite dimensional and central over the rational number field **Q** which is a crossed product for $G$, which means that $D$ contains a maximal subfield $K$ Galois over **Q** with Galois group $G$. Using the fundamental theorem of Albert–Brauer–Hasse–Noether, Schacher [13, 2.1, 2.6] has translated **Q**-admissibility into arithmetic terms: $G$ is **Q**-admissible if and only if there exists a Galois extension $K/\mathbf{Q}$ with Galois group $G(K/\mathbf{Q}) \approx G$ such that for each prime $p$ dividing the order $|G|$ of $G$, there exist at least two distinct primes $q_i = q_i(p)$, $i = 1, 2$, such that

$$[K_{q_i} : \mathbf{Q}_{q_i}]_p = [K : \mathbf{Q}]_p, \qquad i = 1, 2,$$

where $\mathbf{Q}_{q_i}$ is the field of $q_i$-adic rationals, $K_{q_i}$ is the completion of $K$ at any divisor of $q_i$ in $K$, and $[\ :\ ]_p$ denotes the largest power of $p$ dividing $[\ :\ ]$. It follows from this that a necessary group-theoretic condition that $G$ is **Q**-admissible is that $G$ be *Sylow-metacyclic*; i.e., all Sylow subgroups of $G$ are metacyclic [13, th. 4.1]. Schacher then poses the inverse problem: are all Sylow-metacyclic groups **Q**-admissible? Partial results have been obtained by Schacher [13], Gordon and Schacher [5, 6], and Sonn [16, 17]. At present the following (finite) groups are known to be **Q**-admissible: $S_4$, $S_5$ [13], $A_4$ [6], $A_5$ [5],

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SL(2,5) [16], metacyclic groups and Sylow-metacyclic groups possessing a normal 2-complement, e.g. those with cyclic Sylow 2-subgroups [17].

The proof for groups with normal 2-complements [17, th. 2] is based firstly on the result for metacyclic groups and secondly on a theorem of Neukirch on embedding problems with odd order kernels and with prescribed local solutions [12].

This paper is the result of an investigation into Sylow-metacyclic groups with an eye to exploiting Neukirch's theorem as in the case of groups with normal 2-complements, where Neukirch's theorem is applied to reduce the problem to a Sylow 2-subgroup of $G$. As it turns out, Sylow-metacyclic groups are quite well-behaved. The essential idea is to write $G$ as a semidirect product $AN$ with $N$ normal of odd order and $N$ as large as possible, and then to apply Neukirch's theorem in an attempt to reduce the problem from $G$ to $A$.

However, in order to apply Neukirch's theorem, the notion of $\mathbf{Q}$-admissibility needs to be replaced by the notion of *strong* $\mathbf{Q}$-*admissibility*, which is the same as that of $\mathbf{Q}$-admissibility, but with the following modification. Let $n$ be a positive integer given in advance, and let $\mu_n$ denote the group of $n$-th roots of unity. Then in the definition given above for $\mathbf{Q}$-admissibility, we add the requirement that $K \cap \mathbf{Q}(\mu_n) = \mathbf{Q}$.

In the case $G$ solvable, it turns out that $G$ always has a normal $\{2,3\}$-Hall complement, and the problem can be reduced to groups of order $2^a 3^b$. In the case $G$ nonsolvable, $N$ can always be chosen so that $A$ is either simple or very close to it, and the list of such $A$ is quite limited and can be written down; we will do so below. Again the problem can be reduced to such groups $A$. Finally (as will be shown below) (strong) $\mathbf{Q}$-admissibility is preserved under homomorphic images, hence the list of groups $A$ to which the problem can be reduced can be shortened even further. We do not wish to give the impression that the problem is close to being solved completely. For example, it is still not known if the group SL(2,3) of order 24 is $\mathbf{Q}$-admissible, let alone those $A$ which have not yet been realized as Galois groups over $\mathbf{Q}$, for example SL$(2,p^2)$.

## 1. Sylow-metacyclic groups

A metacyclic group is a group having a cyclic normal subgroup whose corresponding factor group is cyclic. A finite group is called a Sylow-metacyclic group if all its Sylow subgroups are metacyclic. This section is devoted to a description of Sylow-metacyclic groups. Our main result will be stated after fixing the notation (see Theorem 1).

NOTATION AND DEFINITIONS. Let $X$ be a finite group and $Y$ and $W$ subgroups of $X$. We will use the following notation:

| | |
|---|---|
| $C_n$ | — the cyclic group of order $n$. |
| $\lvert X \rvert$ | — the order of $X$. |
| $\lvert X : Y \rvert$ | — the index of $Y$ in $X$. |
| $p$ | — will always denote a prime number. |
| $n_p$ | — the highest power of $p$ that divides an integer $n$. |
| $N_Y(W)$ | — the normalizer of $W$ in $Y$. |
| $C_Y(W)$ | — the centralizer of $W$ in $Y$. |
| $S(X)$ | — the largest normal solvable subgroup of $X$. |
| $O(X)$ | — the largest normal subgroup of odd order of $X$. |
| $Z(X)$ | — the center of $X$. |
| $X'$ | — the commutator subgroup of $X$. |
| $M(X)$ | — the Schur multiplier of $X$ [9, p. 628]. |
| $\Phi(X)$ | — the Frattini subgroup of $X$. |
| $\mathrm{Aut}(X)$ | — the automorphism group of $X$. |
| $\mathrm{GF}(q)$ | — the field of $q$ elements. |
| $\mathrm{GL}(n,q)$ | — the group of nonsingular $n \times n$ matrices over $\mathrm{GF}(q)$. |
| $\mathrm{SL}(n,q)$ | — the group of $n \times n$ matrices of determinant 1 over $\mathrm{GF}(q)$. |
| $\mathrm{PGL}(n,q)$ | — $\mathrm{GL}(n,q)$ modulo its center (the scalar matrices). |
| $\mathrm{PSL}(n,q)$ | — $\mathrm{SL}(n,q)$ modulo its center. |
| $\mathrm{PSU}(n,q)$ | — the projective special unitary group [9, p. 233]. |
| $S_n$ | — the symmetric group on $n$ letters. |
| $A_n$ | — the alternating group on $n$ letters. |
| $M_{11}$ | — the Mathieu group of degree 11 [9, p. 154]. |

Let $G$ be a finite group and let $\pi$ be the set of primes dividing $\lvert G \rvert$. Assume that $\pi = \{p_1, p_2, \cdots, p_n\}$ is such that $p_1 < p_2 < \cdots < p_n$. The group $G$ will be said to possess a *Sylow tower* if there exists a normal series in $G$:

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = 1,$$

such that $G_i / G_{i+1}$, $0 \leqq i \leqq n-1$, is isomorphic to a Sylow $p_{i+1}$-subgroup of $G$. Clearly each $G_i$ is a normal Hall subgroup of $G$ and therefore a characteristic subgroup of $G$.

A *stem extension* [8, p. 212] of a group $G$ is a group $E$ having a normal subgroup $A \leqq Z(E) \cap E'$ such that $E/A \simeq G$. Every stem extension of a finite group $G$ is a homomorphic image of a *stem cover* (Darstellungsgruppe [9, p.

630]) of $G$ [8, p. 213]. After listing all Sylow-metacyclic groups $G$ satisfying $H \leqq G \leqq \mathrm{Aut}(H)$, where $H$ is a finite nonabelian simple group, we will need to consider their nontrivial Sylow-metacyclic stem extensions. The following facts will be needed:

1. $\mathrm{PSL}(2, p^n)$, $n = 1$ or 2 and $p \geqq 5$, has a unique stem cover $\mathrm{SL}(2, p^n)$ ([9, p. 646]) which is Sylow-metacyclic.

2. $\mathrm{PSL}(2,9) \simeq A_6$ and $A_7$ each have Schur multiplier of order 6, hence each has 3 nontrivial stem extensions [15]. However, *only one of them is Sylow-metacyclic, namely the one with kernel of order* 2 (for a proof of this fact see Lemma (1.7)). This is $\mathrm{SL}(2,9)$ for $\mathrm{PSL}(2,9)$. That of $A_7$ we will denote by $A_7^+$.

3. The group $M_{11}$ has a trivial Schur multiplier [4, p. 60] and hence no nontrivial stem extension.

4. $\mathrm{PGL}(2, p^n)$, $n = 1$ or 2, $p^n \geqq 5$, has Schur multiplier of order 2 and has two stem covers which we shall denote by $\mathrm{PGL}(2, p^n)^+$ and $\mathrm{PGL}(2, p^n)^-$ [14, p. 122]. Both are Sylow-metacyclic. Since the kernels here have order 2, these are all the nontrivial stem extensions of $\mathrm{PGL}(2, p^n)$.

5. There is a unique Sylow-metacyclic group $\mathrm{PGL}^*(2, p^2) \neq \mathrm{PGL}(2, p^2)$ and lying between $\mathrm{PSL}(2, p^2)$ and $\mathrm{Aut}\,\mathrm{PSL}(2, p^2)$ [1a, p. 20]. It can be shown to have no Sylow-metacyclic stem extensions.

For an explicit discussion of the unique stem covers of $A_6$ and $A_7$ see [15, pp. 242, 245]; for $\mathrm{PGL}(2, p^n)^+$ and $\mathrm{PGL}(2, p^n)^-$ see [14, p. 122].

THEOREM 1. *Let $G$ be a finite Sylow-metacyclic group. Then $G$ is a semidirect product $NA$, where $N$ is a normal subgroup of odd order of $G$ possessing a Sylow tower, and either*

(i) *$G$ is solvable and $A$ is a Hall subgroup of $G$ of order $2^a 3^b$, or*

(ii) *$N = O(G)$ and $A$ is among the following (nonsolvable) groups: $M_{11}$, $A_7$, $A_7^+$, $\mathrm{PSL}(2, p^i)$, $\mathrm{SL}(2, p^i)$, $\mathrm{PGL}(2, p^i)$, $\mathrm{PGL}(2, p^i)^+$, $\mathrm{PGL}(2, p^i)^-$, $i = 1$ or 2, $p^i \geqq 5$, $\mathrm{PGL}^*(2, p^2)$. In this case, if $q$ is a prime divisor of both $|A|$ and $|N|$, then the Sylow $q$-subgroups of $G$ are abelian, and those of both $A$ and $N$ are cyclic. Furthermore, if $A$ is $M_{11}$, $A_7$, $A_7^+$, $\mathrm{PSL}(2, p^i)$ or $\mathrm{SL}(2, p^i)$ (where $A = A'$), then except for the case $A \simeq \mathrm{SL}(2,5)$, $G = A \times N$, and if $A$ is $\mathrm{PGL}(2, p^i)$, $\mathrm{PGL}(2, p^i)^+$ or $\mathrm{PGL}(2, p^i)^-$ or $\mathrm{PGL}^*(2, p^2)$ (where $|A : A'| = 2$), then $|A : C_A(N)| = 1$ or 2.*

NOTE. $A_5 \simeq \mathrm{PSL}(2,4) \simeq \mathrm{PSL}(2,5)$, and $A_6 \simeq \mathrm{PSL}(2,9)$. If $A \simeq \mathrm{SL}(2,5)$ there exist (Frobenius) groups $G = AN$ with $N \simeq C_p \times C_p$, $p \equiv \pm 1 \pmod 5$, such that $A$ acts fixed point free on $N$ [9, p. 500]. For additional details see Lemma (1.9) (c) below.

The theorem will be proved via a series of lemmas, the first two dealing with metacyclic groups.

LEMMA (1.1). (a) *Subgroups and homomorphic images of a metacyclic group are metacyclic.*

(b) *A $p$-group of order $p^3$ and exponent $p$ is not metacyclic.*

(c) *Let $P$ be a $p$-group of order $p^3$, $p$ an odd prime. If $P$ has two subgroups of exponent $p$, $P_1$, and $P_2$, such that $P = P_1P_2$, then $P$ is not metacyclic.*

PROOF. (a) See [9, p. 335].

(b) The only cyclic normal subgroups of such a group are of order $p$. Their factor groups are of order $p^2$ and exponent $p$ and hence not cyclic.

(c) As $P/Z(P)$ is elementary abelian of order 1 or $p^2$ [7, lemma 3.4, p. 11] we use [7, lemma 3.9, p. 183] to get that $(xy)^p = x^p y^p = 1$ for all $x \in P_1$ and $y \in P_2$. Thus $P$ has exponent $p$ and the result follows from part (b).

LEMMA (1.2). *Let $P$ be a metacyclic $p$-group. Let $P_1$ and $P_2$ be nontrivial subgroups of $P$ such that $P = P_1P_2$, $P_1 \triangleleft P$ and $P_1 \cap P_2 = 1$. Then*: (a) *$P_2$ is cyclic, and* (b) *if $p$ is odd then $P_1$ is also cyclic. (The dihedral group of order* 8 *is an example that* (b) *is false if $p = 2$.)*

PROOF. (a) Let $P$ be a counterexample of minimal order. Then $P_2$ is not cyclic. Let $E = \Phi(P_1)P_2$, where $\Phi(P_1)$ is the Frattini subgroup of $P_1$. If $\Phi(P_1) \neq 1$, the minimality of $P$ implies that $P_2$ is cyclic as $|E| < |P|$. Since this is impossible we get that $\Phi(P_1) = 1$ so that $|P_1| = p$ or $P_1$ is elementary abelian of order $p^2$. Then $P_1 \subseteq C_P(P_1) = P_1 \times C_{P_2}(P_1)$ so that $|P/C_P(P_1)| = |P_2 : C_{P_2}(P_1)|$. Since $P/C_P(P_1)$ is isomorphic to a subgroup of $\mathrm{Aut}(P_1) \simeq C_{p-1}$ or $GL(2,p)$, $|P/C_P(P_1)|_p = 1$ or $p$. It follows that $C_{P_2}(P_1) \neq 1$ because $|P_2| \geqq p^2$, and as $C_P(P_1) = P_1 \times C_{P_2}(P_1)$ we get (Lemma 1.1) that $|P_1| = p$ and so $P_2 = C_{P_2}(P_1)$. Therefore $P = P_1 \times P_2$. As $P_2/\Phi(P_2)$ is elementary abelian of order $p^2$ we get that $P/\Phi(P_2)$ is elementary abelian of order $p^3$ contradicting Lemma (1.1).

(b) Suppose that $p$ is odd and that $P_1$ is not cyclic. Then $\bar{P}_1 = P_1/\Phi(P_1)$ is elementary abelian of order $p^2$. Let $Q$ be a subgroup of $P_2$ of order $p$ and set $R = P_1Q$. Let $\bar{R} = R/\Phi(P_1)$ and $\bar{Q} = Q\Phi(P_1)/\Phi(P_1) \simeq Q$. Then $\bar{R} = \overline{P_1}\,\bar{Q}$ is not metacyclic by Lemma (1.1)(c), contradicting Lemma (1.1)(a). Q.E.D.

Since every $p$-subgroup of a group $G$ is contained in a Sylow $p$-subgroup of $G$ and a Sylow $p$-subgroup of a factor group of $G$ is an homomorphic image of a Sylow $p$-subgroup of $G$, we get the following lemma from Lemma (1.1)(a).

LEMMA (1.3). *Subgroups and homomorphic images of a Sylow-metacyclic group are Sylow-metacyclic.*

LEMMA (1.4). *Let $G$ be a finite solvable group whose Sylow 2-subgroups and Sylow 3-subgroups are metacyclic. Then $G$ has a normal $\{2,3\}$-complement; that is, $G = NA$ with $N \lhd G$, $N \cap A = 1$ and $A$ is a Hall subgroup of $G$ of order $2^a 3^b$.*

PROOF. By P. Hall's theorem ([7, p. 231, theorem 4.1]), $G$ contains a Hall $2'$-subgroup, $X$, and a Hall $3'$-subgroup, $Y$. Then $|G| = |X||G|_2 = |Y||G|_3$. Let $P$ be a Sylow $p$-subgroup of $G$, where $p = 2$ or 3. By Lemma (1.1), $P$ has no elementary abelian normal subgroup of order $p^3$, a fact that is denoted by $d_n(P) \leqq 2$. As $|X|$ is odd, $X$ has a normal 3-complement, $C$, and as $(3, |Y|) = 1$, $Y$ has a normal 2-complement, $B$ (see [7, p. 257, theorem 6.1] and [9, p. 437, Satz 5.11]). Since both $C$ and $B$ are Hall $\{2,3\}'$-subgroups of $G$, $C$ is conjugate to $B$ in $G$ [7, p. 231, theorem 4.1]. Hence $N_G(C)$ is conjugate to $N_G(B)$ and therefore $|N_G(B)| = |N_G(C)|$. As $X \subseteq N_G(C)$ we get that $|X| = |G|/|G|_2$ divides $|N_G(C)|$. Since $Y$ contains a Sylow 2-subgroup of $G$ and $Y \subseteq N_G(B)$ we get that $|G|_2$ divides $|N_G(B)| = |N_G(C)|$. As $(|X|, |G|_2) = 1$, we get that $|G| = |X||G|_2$ divides $|N_G(C)|$ and consequently $C \lhd G$. It follows that $C$ is a normal $\{2,3\}$-complement of $G$ so that $G = CA$, $C \cap A = 1$, for any Hall $\{2,3\}$-subgroup $A$, of $G$.

Classification theorems on finite simple groups are applied in the next lemma to determine all finite nonabelian simple Sylow-metacyclic groups.

LEMMA (1.5). *A nonabelian finite simple Sylow-metacyclic group, $G$, is isomorphic to one of the following groups: $A_7$, $M_{11}$, $\mathrm{PSL}(2, p^n)$, $n = 1, 2$, $p^n \neq 2, 3$.*

PROOF. Let $T$ be a Sylow 2-subgroup of $G$. Then $T$ is neither cyclic nor quaternian nor generalized quaternion (see [7, p. 257, theorem 6.1] and [3]). It follows from lemma 1 and theorem 4.10 (ii), p. 199 of [7] that the 2-rank of $T$ is 2. Now the Second Main Theorem of [1] asserts that $G$ is isomorphic to one of the following groups: $\mathrm{PSL}(2,q)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q)$, $A_7$, $M_{11}$, $\mathrm{PSU}(3,4)$, where $q = p^n$, $p$ an odd prime. A Sylow $p$-subgroup $P$ of $\mathrm{PSL}(3,q)$ is isomorphic to the group of all matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where $a$, $b$, $c \in \mathrm{GF}(q)$, the field with $q$ elements. As $|P| = q^3$ and $P$ is of exponent $p$, it follows from Lemma (1.1) that $P$ is not metacyclic. Similarly, a

Sylow $p$-subgroup of $\mathrm{PSU}(3,q)$ is of exponent $p$ and of order $>p^2$ [9, p. 242], hence is not metacyclic.

A Sylow $p$-subgroup of $\mathrm{PSL}(2,q)$ is elementary abelian of order $q = p^n$ and therefore is metacyclic if and only if $n = 1$ or 2. Finally a Sylow 2-subgroup $T$ of $\mathrm{PSU}(3,4)$ is of exponent 4 and order 64 (see [9, p. 242]), thus every factor group $T/T_1$ with $T_1$ cyclic is of exponent 2 or 4 and order at least 16, and hence is not cyclic. Thus, $T$ is not metacyclic. Q.E.D.

Recall that $S(G)$ is the largest normal solvable subgroup of $G$.

LEMMA (1.6). *If $G$ is a nonsolvable finite Sylow-metacyclic group, then $G/S(G)$ is isomorphic to one of the following groups*: $A_7$, $M_{11}$, $\mathrm{PSL}(2,p^n)$, $\mathrm{PGL}(2,p^n)$, $n = 1$ *or* 2, $p^n \geqq 5$, $\mathrm{PGL}^*(2,p^2)$.

PROOF. Let $X = G/S(G)$; then $X$ has no nontrivial normal solvable subgroup. Let $N$ be a minimal normal subgroup of $X$. By Lemma (1.3), $X$ is a Sylow-metacyclic group and by [7, theorem 1.5, p. 17], $N$ is a direct product of isomorphic nonabelian simple groups. As argued in the proof of Lemma (1.5), every nonabelian simple group contains a subgroup isomorphic to $C_2 \times C_2$. Thus $N$ is simple and $Z(N) = C_X(N) \cap N = 1$. As $C_X(N) \lhd X$ we get a direct product $N \times C_X(N)$. If $C_X(N) \neq 1$ then $C_X(N)$ contains a minimal normal subgroup of $X$ which in turn contains a subgroup isomorphic to $C_2 \times C_2$. This contradicts Lemma (1.3). Hence $C_X(N) = 1$ and thus $X$ is isomorphic to a subgroup of $\mathrm{Aut}(N)$.

Lemma (1.5) implies that $N$ is isomorphic to one of the following groups: $A_7$, $M_{11}$, $\mathrm{PSL}(2,p^n)$, $n = 1$ or 2, $p^n \geqq 5$. Now, $\mathrm{Aut}(M_{11}) = M_{11}$ (see [4, p. 60]) and $\mathrm{Aut}(\mathrm{PSL}(2,p)) = \mathrm{PGL}(2,p)$ (see [7], p. 462). Also, $\mathrm{Aut}(A_7) = S_7$ contains the non-metacyclic 2-subgroup $\langle(12)\rangle \times \langle(34)\rangle \times \langle(56)\rangle$ so that $X \neq S_7$. Finally, $\mathrm{Aut}(\mathrm{PSL}(2,p^2)) = \mathrm{PGL}(2,p^2)\cdot C$ where $C \simeq C_2$ and $C$ is generated by a field automorphism. Let $B = \mathrm{PSL}(2,p^2)\cdot C$. Then $C_B(C) = U \times C$ where $U$ contains a subgroup isomorphic to $\mathrm{PSL}(2,p)$ which in turn contains a subgroup isomorphic to $C_2 \times C_2$ (recall that $\mathrm{PSL}(2,3) \simeq A_4$). Thus $B$ is not a Sylow-metacyclic group and as $N \subseteq X \subseteq \mathrm{Aut}(N)$, we are done.

LEMMA (1.7). *Let $Y \lhd X$ with $Y$ cyclic of order* 3 *or* 6, *and $X/Y \simeq$* $\mathrm{PSL}(2,9)$, $A_7$, $\mathrm{PGL}(2,9)$ or $\mathrm{PGL}^*(2,9)$. *Then $X$ is not Sylow-metacyclic.*

PROOF. It suffices to prove this for $X/Y \simeq \mathrm{PSL}(2,9)$, since the other two contain $\mathrm{PSL}(2,9)$ as a subgroup. We may also limit ourselves to the case $|Y| = 3$, because if $|Y| = 6$, we may factor out its subgroup of order 2.

Assume $X$ Sylow-metacyclic. Let $P$ be a Sylow 3-subgroup of $X$. Then $|P| = 27$ and $P/Y$ is a noncyclic group of order 9. If $P$ is nonabelian then $X$ has a normal subgroup $X_1$ with $|X : X_1| = 3$ [9, p. 452, Satz 8.6]. If $Y \not\subseteq X_1$ then $Y \cap X_1 = 1$ and $X = Y \times X_1$ forcing $P$ to be abelian, a contradiction. Hence, $Y \subseteq X_1$ and $1 \neq X_1/Y \lhd X/Y \simeq \mathrm{PSL}(2,9)$ with $|X/Y : X_1/Y| = 3$, again a contradiction. Thus $P$ is abelian so that $P \simeq C_3 \times C_9$.

Note that $N_{X/Y}(P/Y) = N_X(P)/Y$ and as the normalizer of a Sylow 3-subgroup of $\mathrm{PSL}(2,9)$ has order $9\cdot 4$, we get that $|N_X(P)| = 4|P|$. Let $D$ be a subgroup of $N_X(P)$ of order 4; then $DY/Y$ leaves no nontrivial subgroup of $P/Y$ invariant. Let $W$ be the subgroup of $P$ consisting of all elements of $P$ of order 3. Then $W$ is a $D$-invariant elementary abelian group of order 9 with $Y \subset W$. Since $Y$ is a $D$-invariant direct factor of $W$, Maschke's theorem [7, p. 69, theorem 3.2] implies that $W = Y \times Y_1$, where $Y_1$ is a $D$-invariant subgroup of $W$. But then $DY/Y$ leaves the subgroup $Y_1Y/Y$ of $P/Y$ invariant which is impossible as $|Y_1Y/Y| = 3$. Q.E.D.

In the next lemma we show that if $G/S(G)$ is a simple non-abelian Sylow-metacyclic group, then $G$ "almost" splits over $S(G)$.

LEMMA (1.8). *Let $G$ be a nonsolvable finite Sylow-metacyclic group such that $G/S(G)$ is simple. Then $G = S(G)A$ where $A$ is a subgroup of $G$ such that $|S(G) \cap A| \leqq 2$.*

PROOF. Let $G$ be a counterexample of minimal order and let $S = S(G)$. Clearly $S \neq 1$. The proof is divided into steps.

*Step* 1. $G$ contains no proper subgroup $B$ such that $G = SB$ and no proper nonsolvable normal subgroup. In particular $G = G'$.

PROOF. If $B$ is a proper subgroup of $G$ such that $G = SB$, then $G/S \simeq B/S \cap B$ is a nonabelian simple group. It follows that $S(B) = S \cap B$ and $B/S(B)$ is simple with $|B| < |G|$. Thus, $B = S(B)B_1$ with $|S(B) \cap B_1| \leqq 2$ and as $S(B) \cap B_1 = S \cap B_1$ we get that $G = SB = S(S \cap B)B_1 = SB_1$, a contradiction. If $N$ is a normal nonsolvable subgroup of $G$ then $NS/S \lhd G/S$ and as $G/S$ is simple we have that $NS = G$. The above argument implies that $N$ is not a proper subgroup. Finally, as $G'$ is nonsolvable we must have that $G = G'$.

*Step* 2. $S \not\subseteq Z(G)$.

PROOF. Assume that $S \subseteq Z(G)$, then $S \subseteq Z(G) \cap G'$ by Step 1 and so $S$ is isomorphic to a subgroup of $M(G/S)$, the Schur multiplier of $G/S$ (see corollary

11.20, p. 186 in [10]). Now, $G/S$ is one of the groups of Lemma (1.5) and therefore $|M(G/S)| = 2$ unless $G/S \simeq A_7$ or PSL(2,9) in which case $|M(G/S)| = 6$ (see the beginning of this section for the Schur multipliers). Now, Lemma 1.7 implies that $|S| \leqq 2$ and $G = SG$, a contradiction.

*Step* 3. Let $R \neq 1$ be any minimal normal subgroup of $G$ contained in $S$. Then $|R| = r$ or $r^2$, $r$ a prime, and $|S : R| \leqq 2$.

PROOF. Such an $R$ is an elementary abelian group [7, p. 17, theorem 1.5] and so $|R| = r$ or $r^2$, $r$ some prime. Let $\bar{G} = G/R$ and $\bar{S} = S/R$. Then $S(\bar{G}) = \bar{S}$, $\bar{G}/S(\bar{G}) \simeq G/S$ is a simple nonabelian group, and $|\bar{G}| < |G|$. Therefore, there exists a subgroup $U$ of $G$ such that $\bar{G} = (S/R)(U/R)$ where $|S/R \cap U/R| \leqq 2$. Then $G = SU$ and $U$ is nonsolvable. Step 2 now implies that $G = U$ and thus $|S/R \cap U/R| = |S/R| \leqq 2$ as claimed.

*Step* 4. (Final contradiction) If $Q$ is any Sylow subgroup of $S$, then $G = SN_G(Q)$ [7, p. 12, theorem 3.7] and by Step 1 we get that $G = N_G(Q)$. It follows that every Sylow subgroup of $S$ is normal in $G$ and in particular $S$ is nilpotent.

Assume that $|S|$ is even and let $T$ be a Sylow 2-subgroup of $S$. Then $T \lhd G$ and so $T$ contains a minimal normal subgroup of $G$, say $T_1$. By Step 3, $|T_1| = 2$ or 4 and $|S| = 2, 4$ or 8. It is easily verified that Aut$(S)$ is solvable. Since $G/C_G(S)$ is isomorphic to a subgroup of Aut$(S)$ we get that $C_G(S)$ is nonsolvable and as $C_G(S) \lhd G$, Step 1 implies that $C_G(S) = G$. Thus, $S \subseteq Z(G)$, contradicting Step 2. Therefore $|S|$ is odd and by Step 3, $|S| = r$ or $r^2$ where $r$ is an odd prime.

As $S$ is abelian $S \subseteq C_G(S) \lhd G$ so that $C_G(S)/S \lhd G/S$. By Step 2, $C_G(S) \neq G$ so that the simplicity of $G/S$ implies that $C_G(S) = S$. If $(r, |G/S|) = 1$ then $G = SB_2$ with $S \cap B_2 = 1$ by the Schur–Zassenhaus theorem [7, p. 221, theorem 2.1], a contradiction. Hence $r$ divides $|G/S|$. Let $L$ be a Sylow $r$-subgroup of $G$. Since $S \lhd G$ we get that $S \subset L$, $S \neq L$ and $L$ is not abelian as $C_G(S) = S$. A theorem of Huppert [9, p. 452, Satz 8.6] states that $G$ has a normal subgroup of index $r$ which is clearly nonsolvable. This contradicts Step 1.

LEMMA (1.9). *Let $G$ be a nonsolvable Sylow-metacyclic group. Assume that $G = O(G)A$ where $O(G) \cap A = 1$.*

(a) *If* $A \simeq A_7$, $A_7^+$, $M_{11}$, PSL$(2, p^n)$, SL$(2, p^n)$, $n = 1, 2$, $p^n \geqq 5$, *but* $A \not\simeq$ SL(2,5), *then* $G = O(G) \times A$.

(b) *If* $A \simeq$ PGL$(2, p^n)$, PGL$(2, p^n)^+$, PGL$(2, p^n)^-$, $n = 1, 2$, $p^n \geqq 5$, PGL$^*(2, p^2)$, *then* $|A : C_A(O(G))| \leqq 2$.

(c) *If* $A \simeq \mathrm{SL}(2,5)$, *then* $G$ *has a normal* $\{2,3,5\}$*-complement* $B$ *such that* $G = B(O_3 \times O_5 \times A)$ *where* $O_q$ *is a cyclic Sylow* $q$*-subgroup of* $O(G)$, $q = 3,5$.

(d) *In all the above cases* $O(G)$ *has a Sylow tower and if* $q$ *is a prime divisor of* $(|O(G)|, |A|)$ *then the Sylow* $q$*-subgroups of* $G$ *are abelian and the Sylow* $q$*-subgroups of both* $O(G)$ *and* $A$ *are cyclic.*

PROOF. By [7, p. 259] it follows that every Sylow-metacyclic group of odd order has a Sylow tower and thus $O(G)$ has one.

A Sylow $q$-subgroup of $G$, $G_q$, where $q$ is a prime divisor of $(|O(G)|, |A|)$, is of the form $G_q = O_q A_q$ where $O_q$ and $A_q$ are Sylow $q$-subgroups of $O(G)$ and $A$ respectively [9, p. 676, Satz 4.8]. As $O(G) \lhd G$, $O(G) \cap G_q$ is a Sylow $q$-subgroup of $O(G)$ and hence $O_q = O(G) \cap G_q \lhd G_q$. Then $O_q$ and $A_q$ are cyclic by Lemma (1.2) as $O_q \cap A_q = 1$.

Assume now that $A \simeq \mathrm{SL}(2,5)$. As $O(G)$ has a Sylow tower, $O(G)$ contains a normal $\{3,5\}$-complement $B$ which is a normal Hall $\{2,3,5\}'$-subgroup of $G$. Let $\pi = \{2,3,5\}$, then $G$ is $\pi'$-solvable (see [7, p. 226]). Using theorems 3.5 and 3.6, pp. 229–230 of [7] we see that $A$ is contained in a Hall $\pi$-subgroup $G_1$ of $G$. Then $G = BG_1$, $G_1 = (G_1 \cap O(G))A$ and as $A \simeq \mathrm{SL}(2,5)$ we get that $G_1 \cap O(G) = O(G_1)$. Therefore $G = B(O(G_1)A)$ and $|O(G_1)| = 3^a 5^b$. Let $O_3$ (resp. $O_5$) be a Sylow 3 (resp. 5)-subgroup of $O(G)$ contained in $O(G_1)$. Then $O(G_1) = O_5 O_3$ and $O_5 \lhd O(G_1)$, as $O(G_1)$ has a Sylow tower. The group $O_3/C_{O_3}(O_5)$ is isomorphic to a subgroup of $\mathrm{Aut}(O_5)$ whose order is prime to 3 as $O_5$ is cyclic (recall that 5 divides $|A|$). Thus $O_3 = C_{O_3}(O_5)$ and so $O(G_1) = O_5 \times O_3$. For the rest of the proof of (c) we may assume that $|O(G)| = 3^a 5^b$ and prove that $G = O(G) \times A$.

We say that $G$ is of type (a), (b) or (c) if $A$ is one of the groups in (a), (b) or (c) of the statement of the lemma. If $G$ is of type (c) we assume without loss of generality that $|O(G)| = 3^a 5^b$. Let $C = C_A(O(G))$. To complete the proof of the lemma we have to show that $C = A$ if $G$ is of type (a) or (c) and $|A : C| \leqq 2$ if $G$ is of type (b). This will be done by induction on $\sigma(G)$ which is the number of prime divisors of $O(G)$.

Assume first that $\sigma(G) = 1$ so that $O(G)$ is an $r$-group, $r$ an odd prime. If $O(G)$ is cyclic, then $A/C$, which is isomorphic to a subgroup of $\mathrm{Aut}(O(G))$, is abelian. Since the only abelian factor groups of $A$ are of order 1 if $G$ is of type (a) or (c) and 1 or 2 if $G$ is of type (b), we are done. Therefore we may assume that $O(G)$ is not cyclic so that $W = O(G)/\Phi(O(G))$ is an elementary abelian group of order $r^2$. Also $(r, |A|) = 1$ and in particular $G$ is not of type (c). By [7, p. 174, theorem 1.4] $A/C$ is isomorphic to a subgroup of $\mathrm{Aut}(W)$ which is

$GL(2,r)$. Since $p \neq r$, no nonsolvable factor group of $A$ is isomorphic to a subgroup of $GL(2,r)$, except possibly $PSL(2,5) \simeq A_5$, $PGL(2,5) \simeq S_5$, $PGL(2,5)^+$ and $PGL(2,5)^-$ [2, theorem 3.4, p. 153]. The last two groups have center of order 2 with factor group isomorphic to $S_5$ so they are not subgroups of $GL(2,r)$ for $r \neq 5$ [2, lemma 3.3]. If $A_5$ is a subgroup of $GL(2,r)$, $A_5$ would be a subgroup of $SL(2,r)$. This is impossible as $A_5$ has more than one involution while $SL(2,r)$ has one. Hence $PSL(2,5)$ and $PGL(2,5)$ are not subgroups of $GL(2,r)$ so that $A/C$ is solvable. Then $|A:C|=1$ if $G$ is of type (a) and $|A:C| \leqq 2$ if $G$ is of type (b).

Assume now that $\sigma(G)>1$ and let $q$ be the smallest prime divisor of $|O(G)|$. Since $O(G)$ has a Sylow tower, $O(G)=MQ$, where $M$ is a normal $q$-complement of $O(G)$ and $Q$ a Sylow $q$-subgroup of $O(G)$. Then $M \lhd G$. Let $H=MA$. Then $M=O(H)$ and $\sigma(H)=\sigma(G)-1$. By induction we have that $|A:C_A(M)|$ is 1 if $G$ is of type (a) or (c) and 1 or 2 if $G$ is of type (b).

Let $\bar{G}=G/M$, $\bar{Q}=QM/M$, etc. Then $O(\bar{G})=\bar{Q} \simeq Q$. By the case $\sigma(G)=1$ applied to $\bar{G}$, we deduce that $|A:C_A(\bar{Q})| \leqq 2$. Since $PGL(2,p^i)$, resp. $PGL^*(2,p^2)$ has a unique subgroup of index 2, it follows that there is a subgroup $T$ of order 4 contained in both $C_A(\bar{Q})$ and $C_A(M)$. By [7, p. 224, theorem 2.2], $T$ leaves a Sylow $q$-subgroup of $O(G)$ invariant; we may assume that this is $Q$. Since $T$ acts on $Q$ exactly as it acts on $\bar{Q}$, $C_A(Q) \geqq T$. It follows that $T \leqq C = C_A(O(G)) \lhd A$ which forces $[A:C] \leqq 2$, and in cases (a), (c), we have $A=C$. Q.E.D.

Proof of Theorem 1. As mentioned above every Sylow-metacyclic group of odd order has a Sylow tower. Thus, if $G$ is solvable we are done by Lemma (1.4). Hence, we may assume that $G/S(G)$ is nonsolvable.

*Case* 1: *$G/S(G)$ is simple*

Let $S=S(G)$. By Lemma (1.8) there exists a subgroup $A$ of $G$ of minimal order such that $G=SA$ and $|S \cap A| \leqq 2$. As $G/S$ is simple, the minimal choice of $A$ implies that $A=A'$ and consequently $S \cap A \subseteq A' \cap Z(A)$. Thus, $A$ is a stem extension of $A/A \cap S \simeq G/S$. As $G/S$ is one of the groups of Lemma (1.5), the discussion on stem extensions at the beginning of this section implies that $A$ is isomorphic to one of the following groups: $A_7$, $A_7^+$, $M_{11}$, $PSL(2,p^n)$, $SL(2,p^n)$, $n=1,2$, $p^n \geqq 5$. In view of Lemma (1.9) it suffices to show that $G=O(G)A$.

Let $G_2$ be a Sylow 2-subgroup of $G$. Then $G_2=S_2A_2$, where $S_2$ and $A_2$ are Sylow 2-subgroups of $S$ and $A$ respectively (see [9, p. 676, Satz 4.7]). Clearly $S_2 \lhd G_2$ and as $|A_2 \cap S_2| = |A \cap S|_2$ we get that $A \cap S = A_2 \cap S_2 = S \cap A_2$.

Let $D = A \cap S$. We claim that $D \subseteq Z(G_2)$. As this is clearly true if $D = 1$ we may assume that $|D| = 2$. Since $D \subseteq Z(A)$, $D \subseteq Z(A_2)$. It remains to show that $D \subseteq Z(S_2)$. If not, then $Z(S_2) \cap A_2 = 1$. Let $T = Z(S_2)A_2$, $T$ is not metacyclic by Lemma (1.2), a contradiction. Therefore $Z(S_2) \cap A_2 \neq 1$ so that $D = Z(S_2) \cap A_2$ and $D \subseteq Z(S_2) \cap Z(A_2) \subseteq Z(G_2)$ as claimed.

In particular $D \lhd G_2$, $A_2/D$ is not cyclic and since $G_2/D = (S_2/D)(A_2/D)$ is metacyclic we get by Lemma (1.2) that $S_2 = D$. Thus $D$ is a Sylow 2-subgroup of $S$ and as $|D| \leqq 2$, $S$ has a normal subgroup of odd order and of index $|D|$ [7, p. 257, theorem 6.1]. Thus, $S = O(S)D$ and clearly $O(S) = O(G)$ so that $G = SA = O(G)(A \cap S)A = O(G)A$ and we are done.

*Case* 2: *$G/S(G)$ is nonsimple and nonsolvable*

By Lemma (1.6), $G/S \simeq \mathrm{PGL}(2, p^n)$, $n = 1$ or 2, $p^n \geqq 5$, or $\mathrm{PGL}^*(2, p^2)$, and $S = S(G)$. Let $G_1$ be the subgroup of $G$ such that $G_1/S \simeq \mathrm{PSL}(2, p^n)$. As $G_1/S$ is simple we have that $S = S(G_1)$. By the proof of Case 1, $G_1 = O(G_1)A_1$ where $O(G_1) \cap A_1 = 1$ and $|S : O(G_1)| \leqq 2$.

We claim that if a subgroup $A$ of $G$ can be found such that $G = O(G)A$ and $O(G) \cap A = 1$, then we are done. For if such an $A$ exists we get that $O(G) \subseteq S$ so that $O(G) = O(G_1)$ and $G = SA$. It follows that $|A \cap S| \leqq 2$. Then $A \cap S \subseteq Z(A)$, as $A \cap S \lhd A$. If $A \cap S \not\subseteq A'$ then the group $A' \times (A \cap S)$ is a non-Sylow-metacyclic subgroup of $A$, a contradiction. Therefore $A \cap S \subseteq A' \cap Z(A)$ and $A$ is a stem extension of $A/A \cap S \simeq \mathrm{PGL}(2, p^n)$ or $\mathrm{PGL}^*(2, p^2)$. The discussion of stem extensions at the beginning of this section implies that $A \simeq \mathrm{PGL}(2, p^n)$, $\mathrm{PGL}(2, p^n)^+$, $\mathrm{PGL}(2, p^n)^-$ or $\mathrm{PGL}^*(2, p^2)$. Then we are done by Lemma (1.9). Thus in order to complete the proof of Theorem 1 it suffices to prove the following lemma.

LEMMA (1.10). *Let $G$ be a finite Sylow-metacyclic group with $G/S(G) \simeq \mathrm{PGL}(2, p^n)$, $n = 1$ or 2, $p^n \geqq 5$ or $\mathrm{PGL}^*(2, p^2)$. Then there exists a subgroup $A$ of $G$ such that $G = O(G)A$ and $A \cap O(G) = 1$.*

PROOF. Let $G$ be a counterexample of minimal order. Then $O(G) \neq 1$. Let $G_1$ be as above; then $S(G_1) = S$. Let $R$ be a minimal normal subgroup of $G$ contained in $O(G)$. Then $R$ is abelian [7, p. 17, theorem 1.5]. Let $\bar{G} = G/R$; then $S(\bar{G}) = S/R$ and $O(\bar{G}) = O(G)/R$. Since $\bar{G}/S(\bar{G}) \simeq G/S$ and $|\bar{G}| < |G|$, $\bar{G}$ contains a subgroup $E$ such that $\bar{G} = (O(G)/R)(E/R)$ with $O(G) \cap E = R$. It follows that $G = O(G)E = SE$. As $\mathrm{PGL}(2, p^n)$, resp. $\mathrm{PGL}^*(2, p^2)$ contains no nontrivial solvable normal subgroup we have that $E/S \cap E \simeq \mathrm{PGL}(2, p^n)$, resp. $\mathrm{PGL}^*(2, p^2)$ and $S \cap E = S(E)$. Since $|S : O(G_1)| \leqq 2$, $|S : O(G)| \leqq 2$ and thus

$|S \cap E : R| \leqq 2$ and therefore $R = O(E) = E \cap O(G)$. If $|E| < |G|$, then $E = O(E)E_1$ with $O(E) \cap E_1 = 1$ so that $G = O(G)E = O(G)E_1$ with $O(G) \cap E_1 = 1$, a contradiction. Hence $E = G$ and $O(G) = R$ is abelian. Now $G_1$ is a subgroup of index 2 in $G$ which splits over $O(G)$. A theorem of Gaschütz [9, p. 121, Hauptsatz 17.4] implies that $O(G)$ has a complement in $G$, a contradiction.

## 2. A reduction theorem for Q-admissibility

THEOREM 2.1. *Let $G$ be a finite Sylow-metacyclic group written as a semidirect product $AN$, where $N$ is a normal subgroup of odd order of $G$, and either $A$ and $N$ are Hall subgroups of $G$, or $A$ and $N$ are as described in Theorem* 1 *above. Then the strong* $\mathbf{Q}$*-admissibility of $A$ implies that of $G$.*

PROOF. Let $n$ be given; we may assume $n$ divisible by $|N|$. Let $K/\mathbf{Q}$ be a Galois extension with Galois group $A, K \cap \mathbf{Q}(\mu_n) = \mathbf{Q}$, and for each prime $p$ dividing $|A|$ but not $|N|$, let $q_i(p)$, $i = 1, 2$, be primes such that $[K_{q_i} : \mathbf{Q}_{q_i}]_p = [K : \mathbf{Q}]_p$.

Let $p$ now be a prime dividing both $|A|$ and $|N|$, $G_p$ a Sylow $p$-subgroup of $G$. Then $G_p = A_pN_p \simeq A_p \times N_p$, where $A_p$ and $N_p$ are cyclic Sylow $p$-subgroups of $A$ and $N$, respectively, by Theorem 1. Let $|N_p| = p^r$.

Let $\mathcal{A}$ be a generator of $A_p$. Identifying $A$ with $G(K/\mathbf{Q})$, since $K \cap \mathbf{Q}(\mu_{p^r}) = \mathbf{Q}$, $\mathcal{A}$ can be extended to an element $\mathcal{B}$ of $G(K(\mu_{p^r})/\mathbf{Q}(\mu_{p^r}))$. By Chebotarev's density theorem [11, p. 182], there exist infinitely many primes $Q$ of $K(\mu_{p^r})$ whose Frobenius automorphism is $\mathcal{B}$, hence the rational primes $q$ below these $Q$ split completely in $\mathbf{Q}(\mu_{p^r})$ and so by class field theory are congruent to 1 mod $p^r$, and $[K_q : \mathbf{Q}_q]_p = |A_p|$. Fix two such $q = q_i(p)$, $i = 1, 2$.

Next let $p$ divide $|N|$ but not $|A|$. Again $p$ is odd. We claim that there exist infinitely many (a Chebotarev set of) primes $q$ such that $N_p$ is a Galois group over $\mathbf{Q}_p$ and $q$ splits completely in $K$. Indeed, the proof of [17, theorem 1] shows that in order to realize $N_p$ over $\mathbf{Q}_q$ it suffices to realize a semidirect product $N_p^*$ of two cyclic $p$-groups having $N_p$ as factor group. $N_p^*$ has a presentation $x^{p^r} = y^{p^s} = 1, x^{-1}yx = y^t$, where $t \equiv 1 \pmod p$. A sufficient condition that $N_p^*$ be a Galois group over $\mathbf{Q}_q$ is that $q \equiv t \pmod{p^s}$. Identifying $G(\mathbf{Q}(\mu_{p^s})/\mathbf{Q})$ with the group of residues mod $p^s$ prime to $p$, let $\mathcal{C}$ be the automorphism corresponding to $t$. Since $K \cap \mathbf{Q}(\mu_{p^s}) = \mathbf{Q}$, $\mathcal{C}$ can be extended to an element $\mathcal{D}$ of $G(K(\mu_{p^s})/K)$. Thus there exist infinitely many primes $q$ such that $N_p$ is a Galois group over $\mathbf{Q}_q$ and $q$ splits completely in $K$. Fix two such $q = q_i(p)$, $i = 1, 2$.

Finally, for each cyclic subgroup $C$ of $N$, let $q(C)$ be a prime which splits completely in $K(\mu_n)$, again by the density theorem.

We may assume that the $q = q_i(p)$ and $q = q(C)$ are all distinct.

Now consider the global embedding problem given by the epimorphism

$$f : G \to G/N \simeq G(K/\mathbf{Q}).$$

Let $\tilde{\mathbf{Q}}$ denote the algebraic closure of $\mathbf{Q}$, $G_{\mathbf{Q}} = G(\tilde{\mathbf{Q}}/\mathbf{Q})$. A solution to the embedding problem is defined to be a continuous homomorphism

$$g : G_{\mathbf{Q}} \to G$$

such that $fg = \operatorname{res}(\tilde{\mathbf{Q}}/K)$, the canonical restriction map. Since $f$ splits, there is a trivial solution.

Let $S$ be the set of primes $q_i(p)$, $q(C)$ defined above. For each $q \in S$, let $Q$ be a divisor of $q$ in $K$, $\tilde{q}$ a divisor of $Q$ in $\tilde{\mathbf{Q}}$. By a suitable embedding of $\tilde{\mathbf{Q}}$ in $\tilde{\mathbf{Q}}_q$, we may identify $G_{\mathbf{Q}_q}$ with the decomposition subgroup of $\tilde{q}$ in $G_{\mathbf{Q}}$, and $G(K_Q/\mathbf{Q}_q)$ with the decomposition subgroup of $Q$ in $G(K/\mathbf{Q})$. The global problem given by $f$ induces a local one:

$$f_q : G_q \to G(K_q/\mathbf{Q}_q)$$

where $K_q = K_Q$, $G_q = f^{-1}G(K_q/\mathbf{Q}_q)$ and $f_q = f \mid G_q$. A local solution is a homomorphism

$$g_q : G_{\mathbf{Q}_q} \to G_q$$

such that $f_q g_q = \operatorname{res}(\tilde{\mathbf{Q}}_q/K_q)$.

By a theorem of Neukirch [12, p. 148], one may prescribe for each $q \in S$ a local solution $g_q$ at will, and there corresponds a global surjective solution $g : G_{\mathbf{Q}} \to G$ whose restrictions to $G_{\mathbf{Q}_q}$ coincide with the prescribed $g_q$, for each $q \in S$. We proceed to prescribe local solutions in a suitable way.

(1) If $p$ divides $|N|$ but not $|A|$, then for $q = q_i(p)$, $K_q = \mathbf{Q}_q$, $G_q = N$, and $N_p$ is a Galois group over $\mathbf{Q}_q$. Take $g_q$ to be any epimorphism of $G_{\mathbf{Q}_q}$ onto $N_p$.

(2) If $p$ divides both $|N|$ and $|A|$, then for $q = q_i(p)$, $G(K_q/\mathbf{Q}_q)$ is a cyclic Sylow $p$-subgroup of $G(K/Q)$. Since $q \equiv 1 \pmod{|N_p|}$, $G(K_q(q^{1/p^s})/\mathbf{Q}_q) \simeq G_p$, where $p^s = |N_p|$. It follows that there is a local solution $g_q : G_{\mathbf{Q}_q} \to G_q$ such that $f_q g_q = \operatorname{res}(\tilde{\mathbf{Q}}/K_q)$. (Recall $G_p \cong A_p \times N_p$ and $f_q$ maps $N_p$ to 1 and $A_p$ isomorphically onto $G(K_q/\mathbf{Q}_q)$.)

By Neukirch's theorem, then, there exists a global surjective solution $g : G_{\mathbf{Q}} \to G$ satisfying $g(G_{\mathbf{Q}_q}) = G_p$ for $q = q_i(p)$, $i = 1, 2$, for all $p$ dividing $|N|$, and $g(G_{\mathbf{Q}_q}) = C$ for $q = q(C)$, for all cyclic subgroups $C$ of $N$.

Let $L$ be the global solution field (fixed field of the kernel of $g$). Then $G(L/\mathbf{Q}) \simeq G$. By construction, for each $p$ dividing $|G|$, there are primes $q_i(p)$,

$i = 1, 2$, such that $G(L_{q_i}/\mathbf{Q}_{q_i})$ contains a copy of a Sylow $p$-subgroup of $G$. (Note that for $p$ not dividing $|N|$, $L_{q_i} \geqq K_{q_i}$ and the Sylow $p$-subgroups of $G(L/\mathbf{Q})$ are isomorphic to those of $G(K/\mathbf{Q})$.

It remains to show that $L \cap \mathbf{Q}(\mu_n) = \mathbf{Q}$. $g$ induces an isomorphism $g'$ from $G(L/\mathbf{Q})$ to $G$. Let $H$ be the subgroup of $G$ corresponding to $L \cap \mathbf{Q}(\mu_n)$ via $g'$. For each cyclic subgroup $C$ of $N$, $q(C)$ splits completely in $L \cap \mathbf{Q}(\mu_n)$. On the other hand, $q(C)$ has by construction a divisor $Q$ in $L$ with decomposition group corresponding to $C$ via $g'$. Hence $C \subseteq H$, and this holds for every $C$. Hence $N \subseteq H$. This implies $L \cap \mathbf{Q}(\mu_n)$ is contained in $K$ hence in $K \cap \mathbf{Q}(\mu_n) = \mathbf{Q}$. It follows that $G$ is strongly $\mathbf{Q}$-admissible. Q.E.D.

PROPOSITION 2.2. *If a finite group $G$ is (strongly) $\mathbf{Q}$-admissible, then so is every homomorphic image of $G$.*

PROOF. Let $f : G \rightarrow H$ be an epimorphism with kernel $N$. Let $K/\mathbf{Q}$ be Galois with group $G$, and, for each $p$ dividing $|G|$, let $q_i = q_i(p)$, $i = 1, 2$, be primes such that $|G|_p = [K_{q_i} : \mathbf{Q}_{q_i}]_p$, $i = 1, 2$. Let $K'$ be the subfield of $K$ corresponding to $N$. Then $[K_{q_i} : K'_{q_i}]_p \leqq [K : K']_p$ and $[K_{q_i} : \mathbf{Q}_{q_i}]_p \leqq [K' : \mathbf{Q}]_p$. Therefore

$$|G|_p = [K_{q_i} : \mathbf{Q}_{q_i}]_p = [K_{q_i} : K'_{q_i}]_p [K'_{q_i} : \mathbf{Q}_{q_i}]_p$$

$$\leqq [K : K']_p [K' : \mathbf{Q}]_p = [K : \mathbf{Q}]_p = |G|_p,$$

so equality holds throughout; in particular $[K'_{q_i} : \mathbf{Q}_{q_i}]_p = [K' : \mathbf{Q}]_p$, $i = 1, 2$. Q.E.D.

As an immediate consequence of Theorems 1, 2.1 and Proposition 2.2 above, we obtain a group-theoretic reduction of the strong $\mathbf{Q}$-admissibility problem.

THEOREM 2.3. *In order to prove that all finite Sylow-metacyclic groups are strongly $\mathbf{Q}$-admissible, it remains to prove it for the following ones:*

(a) *groups of order* $2^a 3^b$,
(b) $\mathrm{SL}(2, p)$, *$p$ a prime* $\geqq 5$,
(c) $\mathrm{SL}(2, p^2)$, *$p$ a prime* $\geqq 3$,
(d) $\mathrm{PGL}(2, p)^+$ *and* $\mathrm{PGL}(2, p)^-$, *$p$ a prime* $\geqq 5$,
(e) $\mathrm{PGL}(2, p^2)^+$, $\mathrm{PGL}(2, p^2)^-$, *and* $\mathrm{PGL}^*(2, p^2)$, *$p$ a prime* $\geqq 3$,
(f) $A_7^+$,
(g) $M_{11}$ *(Mathieu group).*

REMARKS. (1) The groups in (b), (c), (f), and (g) in Theorem 2.3 have no abelian factor groups. It follows that for these groups, $\mathbf{Q}$-admissibility implies strong $\mathbf{Q}$-admissibility. The groups in (d) and (e) have abelian factor groups of orders 1 and 2 only. Thus for them, $\mathbf{Q}$-admissibility by means of a field $K$ not containing the cube roots of unity is as good as strong $\mathbf{Q}$-admissibility. Thus the $\mathbf{Q}$-admissibility of nonsolvable Sylow-metacyclic groups is virtually (but not quite) reduced to that of the groups (b)–g).

(2) $SL(2,5)$ is $\mathbf{Q}$-admissible [16, theorem 4]. Consequently, *every finite Sylow-metacyclic group having $A_5$ as a factor group is $\mathbf{Q}$-admissible.*

(3) Every finite metacyclic group is $\mathbf{Q}$-admissible [17, theorem 1]. In order to apply this result, we must prove it for strong $\mathbf{Q}$-admissibility.

THEOREM 2.4. *Every finite metacyclic group is strongly $\mathbf{Q}$-admissible.*

PROOF. We must modify the proof of [17, theorem 1] so that $K \cap \mathbf{Q}(\mu_r) = \mathbf{Q}$, where $r$ is given in advance. In the following discussion, we refer to the proof of [17, theorem 1].

First, the reduction to semidirect products of two cyclic groups clearly carries over. We may also assume that $r$ is divisible by $|G| = mn$. Thus the field $T$ can be chosen so that $T \cap \mathbf{Q}(\mu_r) = \mathbf{Q}$.

Now in addition to the set of $N$ primes $p$ which remain prime in $T$ and are congruent to $t$ mod $n$, we choose a prime $p_0$ which splits completely in $T(\mu_r)$ and add it to the set $S$. For $p = p_0$, we have $T(\mu_r)_p = \mathbf{Q}_p$, hence $G'_p = 1$. Hence the cohomology map

$$H^1(G_{\mathbf{Q}}, Y) \to \prod_{p \in S} H^1(G_{\mathbf{Q}_p}, Y)$$

is still surjective, and therefore there is a Galois extension $K/\mathbf{Q}$ satisfying the same conditions as before and in addition, $K_p/\mathbf{Q}_p$ is cyclic of degree $n$ (unramified, for example), for $p = p_0$.

Now let $k = K \cap \mathbf{Q}(\mu_r)$. Then $k \cap T \leqq T \cap \mathbf{Q}(\mu_r) = \mathbf{Q}$. $p_0$ splits completely in $k$ and in $T$, hence in $kT$. But then $n = [K_{p_0} : \mathbf{Q}_{p_0}] \leqq [K_{p_0} : (kT)_{p_0}] \leqq [K : kT]$. Since $n = [K : T]$, we have $kT = T$, $k = \mathbf{Q}$, Thus $G$ is strongly $\mathbf{Q}$-admissible. Q.E.D.

By Theorem 2.1 above, we have

COROLLARY 2.5. *Every finite solvable Sylow-metacyclic group whose $\{2,3\}$-Hall subgroups are metacyclic is $\mathbf{Q}$-admissible.*

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